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SMOOTH NORMAL APPROXIMATIONS OF EPI-LIPSCHITZIAN SUBSETS OF $\mathbb{R}^n$*

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Abstract. A sequence (M_k) of closed subsets of $\mathbb{R}^n$ converges normally to $M \subset \mathbb{R}^n$ if (sc) $M = \limsup M_k = \liminf M_k$ in the sense of Painlevé–Kuratowski and (nc) $\limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M)$, where $\mathrm{G}(N_M)$ (resp., $\mathrm{G}(N_{M_k})$) denotes the graph of N_M (resp., N_{M_k}), Clarke's normal cone to M (resp., M_k).

This paper studies the normal convergence of subsets of $\mathbb{R}^n$ and mainly shows two results. The first result states that every closed epi-Lipschitzian subset M of $\mathbb{R}^n$, with a compact boundary, can be approximated by a sequence of smooth sets (M_k), which converges normally to M and such that the sets M_k and M are lipeomorphic for every k (i.e., the homeomorphism between M and M_k and its inverse are both Lipschitzian). The second result shows that, if a sequence (M_k) of closed subsets of $\mathbb{R}^n$ converges normally to an epi-Lipschitzian set M, and if we additionally assume that the boundary of M_k remains in a fixed compact set, then, for k large enough, the sets M_k and M are lipeomorphic.

In Cornet and Czarnecki [*Cahier Eco-Maths 95-55,* 1995], direct applications of these results are given to the study (existence, stability, etc.) of the generalized equation $0 \in f(x^*) + N_M(x^*)$ when M is a compact epi-Lipschitzian subset of $\mathbb{R}^n$ and $f : M \to \mathbb{R}^n$ is a continuous map (or more generally a correspondence).

Key words. epi-Lipschitzian, normal convergence, smooth approximation, lipeomorphism, homeomorphism, Clarke's normal cone

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1. Introduction.

A closed subset M of $\mathbb{R}^n$ is said to be epi-Lipschitzian if its Clarke's normal cone $N_M(x)$ is pointed (i.e., if $N_M(x) \cap -N_M(x) = \{0\}$) at every $x \in M$. This class of sets, introduced in optimization by Rockafellar [16], is of particular importance since it includes both (i) closed convex sets with a nonempty interior and (ii) sets defined by finite smooth inequality constraints satisfying a nondegeneracy assumption (independence of the binding constraints). Closed epi-Lipschitzian subsets M of $\mathbb{R}^n$ are equivalently defined as sets that can be locally written as the epigraph of a Lipschitzian function (see [16]).

A sequence (M_k) of closed subsets of $\mathbb{R}^n$ converges normally to $M \subset \mathbb{R}^n$ if (sc) $M = \limsup M_k = \liminf M_k$ in the sense of Painlevé–Kuratowski and (nc) $\limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M)$, where $\mathrm{G}(N_M) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n | x \in M, y \in N_M(x)\}$ (resp., $\mathrm{G}(N_{M_k})$) denotes the graph of N_M (resp., N_{M_k}), Clarke's normal cone to M (resp., M_k).

This paper studies the normal convergence of subsets of $\mathbb{R}^n$ and mainly shows two results. The first result (Theorem 2.1) states that every closed epi-Lipschitzian subset M of $\mathbb{R}^n$, with a compact boundary, can be approximated by a sequence of smooth sets (M_k), which converges normally to M and such that, for every k, the sets M_k and M are lipeomorphic (i.e., the homeomorphism between M and M_k and its

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inverse are both Lipschitzian). Moreover, we prove that one can additionally assume that the approximating sequence (M_k) is internal (resp., external) in the following sense: $M_k \subset \mathrm{int}M_{k+1}$ (resp., $M_{k+1} \subset \mathrm{int}M_k$) for all $k \in \mathbb{N}$. This result extends previous ones in the literature (which do not consider the lipeomorphism properties); see Benoist [1] and (with a different formalism, without the geometrical concept of Clarke's cones) Nečas [15] (in Russian), Massari and Pepe [13], and Doktor [8].

In the above result, the lipeomorphism property is in fact a consequence of the normal convergence of the sequence (M_k). This is a consequence of our second result (Theorem 2.2), which states that if (M_k) is a sequence of closed subsets of $\mathbb{R}^n$ which converges normally to an epi-Lipschitzian set M, then the sets M_k and M are lipeomorphic for k large enough if we additionally assume that, for all k, $\mathrm{bd}M_k$ remains in some given compact set $K \subset \mathbb{R}^n$. In fact, we shall show (Theorem 2.3) that one can weaken assertion (nc) by only assuming that the convex hull of the set $\{p \in \mathbb{R}^n | (x,p) \in \limsup \mathrm{G}(N_{M_k})\}$ is pointed for every $x \in M$.

In [7], direct applications of these results are given to the study (existence, stability, etc.) of the generalized equation $0 \in f(x^*) + N_M(x^*)$ when M is a compact epi-Lipschitzian subset of $\mathbb{R}^n$ and $f : M \to \mathbb{R}^n$ is a continuous map (or more generally a correspondence).

The paper is organized as follows. The definitions and the main results are given in section 2. The proof of the approximation result (Theorem 2.1) is given in section 3, and the proof of the lipeomorphism result (Theorem 2.3) is given in section 4.

2. Definitions and statement of the results.

2.1. Preliminaries[1]**.** Let M be a closed subset of $\mathbb{R}^n$ and let $x \in M$. We define Clarke's normal cone to M at x, denoted $N_M(x)$, in two steps as follows. We first call perpendicular vector to M at x every vector in the set

$$\perp_M(x) = \{v \in \mathbb{R}^n | \exists \alpha > 0, B(x + \alpha v, \alpha\|v\|) \cap M = \emptyset\}.$$

Then Clarke's normal cone to M at x is the closure of the convex hull of the following limiting cone:

$$\hat{N}_M(x) = \{v \in \mathbb{R}^n | \exists (x_k)_{k\in\mathbb{N}} \subset M,\ \forall k \in \mathbb{N},\ \exists v_k \in \perp_M(x_k),\ (x_k) \to x, (v_k) \to v\}.$$

We now define Clarke's tangent cone to M at x, denoted $T_M(x)$, as the negative polar cone of $N_M(x)$, i.e.,

$$T_M(x) = \{u \in \mathbb{R}^n \mid \forall v \in N_M(x), (u|v) \le 0\}.$$

We recall that a closed subset M of $\mathbb{R}^n$ is said to be epi-Lipschitzian if $N_M(x)$ is pointed (i.e., $N_M(x) \cap -N_M(x) = \{0\}$) for all $x \in M$. We say that M is C^k-smooth if it is a C^k (with $k \in \{1, \dots, \infty\}$) submanifold with a boundary of $\mathbb{R}^n$ of full dimension,

[1] We let $\mathbb{R}_+ = \{x \in \mathbb{R} | x \ge 0\}$. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ belong to $\mathbb{R}^n$, we denote the scalar product of $\mathbb{R}^n$ by $(x|y) = \sum_{i=1}^n x_i y_i$, the Euclidean norm by $\|x\| = \sqrt{(x|x)}$; we denote $B(x,r) = \{y \in \mathbb{R}^n | \ \|x-y\| < r\}$, $\overline{B}(x,r) = \{y \in \mathbb{R}^n | \ \|x-y\| \le r\}$, $S(x,r) = \{y \in \mathbb{R}^n | \ \|x-y\| = r\}$, $\overline{B} = \overline{B}(0,1)$, and $S = S(0,1)$. If $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^n$, and $x \in \mathbb{R}^n$, we let $d(x,X) = \inf_{y\in X} \|x-y\|$ (also denoted $d_X(x)$), and we denote by $X + Y = \{x + y | x \in X, y \in Y\}$ the sum of X and Y, $B(X,r) = X + B(0,r)$, $\mathrm{cl}X$ or $\overline{X}$ the closure of X, $\mathrm{int}X$ the interior of X, $\mathrm{bd}X$ the boundary of X, and $\mathrm{co}X$ the convex hull of X. If X and Y are two nonempty compact subsets of $\mathbb{R}^n$, $\delta(X,Y) = \max\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\}$ is the Hausdorff distance between X and Y. A correspondence Φ from $X \subset \mathbb{R}^n$ to $\mathbb{R}^m$ is a map from X to the set of all the subsets of $\mathbb{R}^m$ and the graph of Φ, denoted $\mathrm{G}(\Phi)$, is defined by $\mathrm{G}(\Phi) = \{(x,y) \in X \times \mathbb{R}^m | y \in \Phi(x)\}$.

i.e., if for all $\overline{x} \in M$, there is an open neighborhood U of $\overline{x}$ and a C^k function $f : U \to \mathbb{R}$ such that $M \cap U = \{x \in U | f(x) \leq 0\}$, and such that $\nabla f(x) \neq 0$ if $f(x) = 0$. M is said to be smooth if it is C^∞-smooth.

We recall the following definitions and properties associated with a sequence (M_k) of subsets of $\mathbb{R}^n$ (see, for example, Kuratowski [12]):

$$\begin{aligned}\liminf M_k &= \{x \in \mathbb{R}^n | \exists (x_k) \subset \mathbb{R}^n, x_k \to x, x_k \in M_k \text{ for all } k\};\\ \limsup M_k &= \{x \in \mathbb{R}^n | \exists (x_k) \subset \mathbb{R}^n, \exists \varphi \in \mathcal{I}, x_k \to x, x_k \in M_{\varphi(k)} \text{ for all } k\},^2\end{aligned}$$

where $\mathcal{I}$ is the set of all increasing maps $\varphi : \mathbb{N} \to \mathbb{N}$.

We recall that the inclusion $\liminf M_k \subset \limsup M_k$ always holds true; the sequence (M_k) is said to be set-convergent if $\liminf M_k = \limsup M_k$. We say that the sequence (M_k) is smooth if the set M_k is smooth for k large enough. We say that it is increasing (resp., decreasing) if $M_k \subset \mathrm{int} M_{k+1}$ (resp., $M_{k+1} \subset \mathrm{int} M_k$) for all $k \in \mathbb{N}$. If (M_k) is an increasing (resp., decreasing) sequence, then one notices that it set-converges to some $M \subset \mathbb{R}^n$ if and only if $M = \mathrm{cl}(\cup_{k\in\mathbb{N}} M_k)$ (resp., $M = \cap_{k\in\mathbb{N}} M_k$). An increasing (resp., decreasing) converging sequence is also called an internal (resp., external) approximation of its set-limit.

2.2. Statement of the results. We give a stronger notion of set-convergence which involves both the set-convergence and the convergence of the graph of the normal cones in the following sense.

DEFINITION 2.1. *We say that a sequence (M_k) of closed subsets of $\mathbb{R}^n$ is a normal approximation of a closed subset $M \subset \mathbb{R}^n$ (or converges normally to M) if the two following assertions hold:*

(sc) *(set convergence)* $M = \limsup M_k = \liminf M_k$;
(nc) *(normal convergence)* $\limsup \mathrm{G}(N_{M_k}) \subset \mathrm{G}(N_M)$.

Remark 2.1 (the convex case). Let (M_k) be a sequence of closed convex subsets of $\mathbb{R}^n$. Assume that (M_k) set-converges to some subset $M \subset \mathbb{R}^n$. Then one easily notices that the set M is convex and that (M_k) is a normal approximation of M.

The next theorem shows the existence of internal and external smooth normal approximations of a compact epi-Lipschitzian subset of $\mathbb{R}^n$, which satisfy additional properties also of interest for themselves (in fact we weaken the compactness assumption by assuming only that M is closed and that $\mathrm{bd}M$ is compact). We recall that subsets M and N of $\mathbb{R}^n$ are lipeomorphic if there exists a map $\Phi : M \to N$ which is a lipeomorphism, i.e., is a Lipschitzian invertible map with a Lipschitzian inverse.

THEOREM 2.1. *Let M be a closed epi-Lipschitzian subset of $\mathbb{R}^n$, such that $\mathrm{bd}M$ is nonempty and compact. Then there exists a smooth internal normal approximation and a smooth external normal approximation of M which both additionally satisfy the following properties:*

(lip) *the sets M_k and M are lipeomorphic for all k;*
(lip^c) *the sets $\mathbb{R}^n \setminus \mathrm{int} M_k$ and $\mathbb{R}^n \setminus \mathrm{int} M$ are lipeomorphic for all k;*
(L) *there is $\ell > 0$ and a compact subset $K \subset \mathbb{R}^n$ such that, for all k, $\mathrm{bd}M_k \subset K$ and*

$$\delta(\mathrm{bd}M_k, \mathrm{bd}M_{k+1}) \leq \ell \min\{\|x - y\| \,|x \in \mathrm{bd}M_k, y \in \mathrm{bd}M_{k+1}\}.$$

Theorem 2.1 is proved in section 3. A general discussion about assertion (L) is given in the next section.

[2] Equivalently, $\limsup M_k = \cap_{p\in\mathbb{N}} \mathrm{cl}(\cup_{k\geq p} M_k)$. Note also that for every subsequence $(M_{\varphi(k)})$ of (M_k), one has that $\liminf M_k \subset \liminf M_{\varphi(k)} \subset \limsup M_{\varphi(k)} \subset \limsup M_k$.

At this stage, it is worth pointing out that in Theorem 2.1 the two lipeomorphism assertions (lip) and (lip^c) are a consequence of the normal convergence of the sequence (M_k) as is shown in the next result.

THEOREM 2.2. *Let (M_k) be a sequence of closed subsets of $\mathbb{R}^n$ such that the boundaries $\mathrm{bd}M_k$ remain in a given compact subset $K \subset \mathbb{R}^n$. Assume that (M_k) set-converges to some subset $M \subset \mathbb{R}^n$ and that*

($*$) *(M_k) converges normally to M, and M is epi-Lipschitzian.*

Then, for k large enough,

(i) *the sets M, $\mathbb{R}^n \setminus \mathrm{int}M$, M_k and $\mathbb{R}^n \setminus \mathrm{int}M_k$ are epi-Lipschitzian;*

(ii) *M_k is lipeomorphic to M and $\mathbb{R}^n \setminus \mathrm{int}M_k$ is lipeomorphic to $\mathbb{R}^n \setminus \mathrm{int}M$.*

Theorem 2.2 is a consequence of the following result, which slightly weakens assertion ($*$) by noticing that the condition (nc) of normal convergence implies that

$$\{p \in \mathbb{R}^n | (x,p) \in \limsup \mathrm{G}(N_{M_k})\} = \limsup_{x' \to x, k \to \infty} N_{M_k}(x') \subset N_M(x).$$

THEOREM 2.3. *Theorem 2.2 remains true if one replaces assertion ($*$) with the following assertion:*

($**$) $\mathrm{co}\,\limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ *is pointed for all $x \in M$.*

The proof of Theorem 2.3 is given in section 4.

Remark 2.2. Theorem 2.2 and Theorem 2.3 may not be true if $\mathrm{bd}M_k$ does not remain in a fixed compact set K. Consider $M = \mathbb{R} \times \mathbb{R}_+$ and the smooth internal normal approximation of M defined by $M_k = \mathbb{R} \times [1/k, \infty) \setminus B((k, 3/k), 1/k)$ for $k \geq 1$. Then, for every k, M_k is clearly not homeomorphic to M.

Remark 2.3. Theorem 2.2 may not be true if the set-limit M is not epi-Lipschitzian. Consider $M = \{0\}$ in $\mathbb{R}$ and the smooth external normal approximation of M defined by $M_k = [-1/k, 1/k]$. Then for every k, M_k is not homeomorphic to M. Similarly, Theorem 2.3 may not be true without the assumption that $\mathrm{co}\,\limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ is pointed for all $x \in M$, even if one assume that M is epi-Lipschitzian. Consider $M = [1, 1]$ in $\mathbb{R}$ and the smooth internal normal approximation of M defined by $M_k = [-1, -1/k] \cup [1/k, 1]$, and notice that $\mathrm{co}\,\limsup_{x' \to 0, k \to \infty} N_{M_k}(x') = \mathbb{R}$.

Remark 2.4. Theorem 2.3 may not be true if we do not assume that $\limsup M_k = \liminf M_k$. Consider $M_{2k} = \overline{B}(0,1)$, $M_{2k+1} = \overline{B}(0,1) \setminus B(0, 1/2)$ (or $M_{2k+1} = \emptyset$), and notice that $\overline{B}(0,1) = \limsup M_k$.

2.3. General remarks about assertion (L).

Remark 2.5. There may exist a normal (external or internal) approximation satisfying all the conclusions of Theorem 2.1 except for assertion (L). Consider the subset $M = [0,1]$ of $\mathbb{R}$ and the sets $M_k = [-1/k, 1 + 1/2^k]$.

Remark 2.6. If $\mathrm{bd}M$ is not compact, there may not exist an internal (or an external) normal approximation of M which satisfies assertion (L). Consider the following closed epi-Lipschitzian subset of $\mathbb{R}$:

$$M = \left(\mathbb{R}_- \setminus \cup_{k=1}^{\infty} \left(-k - \frac{1}{k+1}, -k + \frac{1}{k+1}\right)\right) \cup \left(\cup_{k=1}^{\infty} \left[k - \frac{1}{k+1}, k + \frac{1}{k+1}\right]\right).$$

Then, if (M_k) is any smooth internal (or external) normal approximation of M, we let the reader check that it does not satisfy assertion (L).

Remark 2.7. Note that the inequality

$$\delta(X,Y) \geq \min\{\|x - y\| \, | x \in X, y \in Y\}$$

is always true if X and Y are two nonempty compact subsets of $\mathbb{R}^n$. Hence, one necessarily has $\ell \geq 1$ in assertion (L) of Theorem 2.1.

Remark 2.8. If we additionally assume that (M_k) is increasing or decreasing, then

$$(\mathrm{L}') \qquad \forall k,\ \delta(\mathrm{bd} M_k, \mathrm{bd} M) \leq \ell \min\{\|x-y\| \,|\, x \in \mathrm{bd} M, y \in \mathrm{bd} M_k\}.$$

Indeed, one just needs to notice that $\delta(\mathrm{bd} M_k, \mathrm{bd} M) \leq \sum_{i=k}^{\infty} \delta(\mathrm{bd} M_i, \mathrm{bd} M_{i+1})$ and that

$$\min\{\|x-y\| \,|\, x \in \mathrm{bd} M, y \in \mathrm{bd} M_k\} \geq \sum_{i=k}^{\infty} \min\{\|x-y\| \,|\, x \in \mathrm{bd} M_i, y \in \mathrm{bd} M_{i+1}\}.$$

Assertion (L$'$) is no longer true if we do not assume that the sequence (M_k) is increasing or decreasing. Consider the set $M = [0,1]$, the sets $M_{2k} = [-1/k, 1]$ and $M_{2k+1} = [0, 1 + 1/(k+1)]$ for all $k \geq 1$. Then (M_k) is a smooth approximation of M which satisfies assertion (L), but the above property (L$'$) is not true.

2.4. Other concepts of normal convergence.

2.4.1. Involving the subdifferential of the distance function. We first recall the definition of Clarke's subdifferential of a locally Lipschitzian function.[3] Let U be an open subset of $\mathbb{R}^n$ and consider $f : U \to \mathbb{R}$; if f is differentiable at $x \in U$, we denote $\nabla f(x)$ the gradient of f at x. If f is locally Lipschitzian, its subdifferential $\partial f(x)$ at $x \in U$ is defined by

$$\partial f(x) = \mathrm{co}\{\lim_{k\to\infty} \nabla f(x_k) | x_k \to x,\ x_k \in \mathrm{Dom}(\nabla f)\},$$

where $\mathrm{Dom}(\nabla f)$ is the set on which f is differentiable. In the case of the distance function d_M to a closed set $M \subset \mathbb{R}^n$, one can be more precise. Indeed, from Clarke [4, Thm. 2.5.6],

$$(1) \qquad \partial d_M(x) = \mathrm{co}\Big((\hat{N}_M(x) \cap S) \cup \{0\} \Big),$$

where $\hat{N}_M(x)$ is the limiting normal cone defined previously and S is the unit sphere in $\mathbb{R}^n$.

It seems natural to compare the normal convergence with the following concept of ∂-convergence, in which one replaces the normal cone by the subdifferential of the distance function. More precisely, we say that a sequence (M_k) of closed subsets of $\mathbb{R}^n$ ∂-converges to a closed subset $M \subset \mathbb{R}^n$ if it satisfies assertion (sc) together with

$$(\partial\mathrm{c}) \quad \limsup \mathrm{G}(\partial d_{M_k}) \subset \mathrm{G}(\partial d_M).$$

It is worth noticing that the ∂-convergence can be formulated only in terms of the distance function, by noticing that assertion (sc) can be equivalently reformulated as follows:

$$(\mathrm{sc}') \quad \forall x \in \mathbb{R}^n,\ \lim_{k\to\infty} d_{M_k}(x) = d_M(x).$$

The link between normal convergence and ∂-convergence can be summarized as follows. It will appear that the concept of ∂-convergence is too strong (for a matter of normalization), even in the epi-Lipschitzian case. Indeed, in this paper we

[3] If $X \subset \mathbb{R}^n$, a map $f : X \to \mathbb{R}^m$ is locally Lipschitzian if, for all $x \in X$, there is a neighborhood U of x and a real number $K \geq 0$ such that $\|f(y) - f(z)\| \leq K\|y - z\|$ for all y and z in U.

show that every compact epi-Lipschitzian set can be approximated in the sense of normal convergence by a sequence of smooth sets. This result is no longer true with the ∂-convergence as shown below (Proposition 2.2) by taking $M = \mathbb{R}^2 \setminus \mathrm{int}\mathbb{R}^2_+$. Furthermore, in the epi-Lipschitzian case, the following proposition shows that the ∂-convergence implies the normal convergence, a result which is no longer true in general (see Remark 2.9).

PROPOSITION 2.1. *Let (M_k) be a sequence of closed subsets of $\mathbb{R}^n$ which ∂-converges to some epi-Lipschitzian set $M \subset \mathbb{R}^n$. Then (M_k) converges normally to the set M.*

Proof of Proposition 2.1. Let $(x, p) \in \limsup \mathrm{G}(N_{M_k})$. Then there is a sequence (x_k) converging to x, a sequence (p_k) converging to p, and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$, such that $x_k \in M_{\varphi(k)}$ and $p_k \in N_{M_{\varphi(k)}}(x_k)$ for all k. Since $N_{M_{\varphi(k)}}(x_k) = \mathrm{cl}[\cup_{\lambda\geq 0}\lambda\partial d_{M_{\varphi(k)}}(x_k)]$ (see Clarke [4]) for all k, there is a sequence $(\lambda^k_r)_{r\in\mathbb{N}}$ in $\mathbb{R}_+$ and a sequence $(v^k_r)_{r\in\mathbb{N}}$ in $\partial d_{M_{\varphi(k)}}(x_k)$ such that $\lambda^k_r v^k_r$ converges to p_k when $r \to \infty$. Hence, without any loss of generality (using a diagonal argument), we may assume that $p = \lim_{k\to\infty} \lambda_k v_k$, with $\lambda_k \geq 0$ and $v_k \in \partial d_{M_{\varphi(k)}}(x_k) \subset \overline{B}(0, 1)$, and that the bounded sequence (v_k) converges to some $v \in \mathbb{R}^n$. Then, from above and (1), for every integer k,

$$v_k \in \partial d_{M_{\varphi(k)}}(x_k) = \mathrm{co}\Big((\hat{N}_{M_{\varphi(k)}}(x_k) \cap S) \cup \{0\}\Big).$$

Hence, from Carathéodory's theorem, there are $n+1$ elements (v^i_k, λ^i_k) ($i \in \{1, \dots, n+1\}$) in $\mathbb{R}^n \times \mathbb{R}_+$ such that $v^i_k \in \hat{N}_{M_{\varphi(k)}}(x_k) \cap S \subset \partial d_{M_{\varphi(k)}}(x_k)$, $\sum_{i=1}^{n+1} \lambda^i_k = 1$, and $\mu_k \in [0, 1]$ such that

$$v_k = \mu_k \sum_{i=1}^{n+1} \lambda^i_k v^i_k.$$

Again, without any loss of generality, we may assume that $(v^1_k, \dots, v^{n+1}_k, (\lambda^1_k,\dots, \lambda^{n+1}_k), \mu_k)$ converges to some element $(v^1, \dots, v^{n+1}, (\lambda^1,\dots,\lambda^{n+1}), \mu) \in S^{n+1} \times \Sigma \times [0, 1]$, where Σ is the unit simplex of $\mathbb{R}^{n+1}$. From assertion (∂c), we get that $v \in \partial d_M(x)$ and that $v^i \in \partial d_M(x)$ for all $i \in \{1, \dots, n + 1\}$. But for all i, from above $v^i \in S$ and $v^i \in \mathrm{co}\Big((\hat{N}_M(x) \cap S) \cup \{0\}\Big)$, noticing that

$$\mathrm{co}\Big((\hat{N}_M(x) \cap S) \cup \{0\}\Big) \cap S = \hat{N}_M(x) \cap S,$$

we deduce that $v^i \in \hat{N}_M(x) \cap S$. Then $w = \sum_{i=1}^{n+1} \lambda^i v^i \in \mathrm{co}(\hat{N}_M(x) \cap S)$, which does not contain 0 since M is epi-Lipschitzian; hence $w \neq 0$. Recalling that $p = \lim_{k\to\infty} \lambda_k v_k = \lim_{k\to\infty} \lambda_k\mu_k \sum_{i=1}^{n+1} \lambda^i_k v^i_k$, the sequence $(\lambda_k\mu_k)$ converges to $\rho = \|p\|/\|w\|$, and $p = \rho w$ with $w \in \mathrm{co}(\hat{N}_M(x) \cap S) \subset \partial d_M(x)$; hence $p \in N_M(x)$. This shows that (M_k) converges normally to M. □

Remark 2.9. Proposition 2.1 may no longer be true if M is not epi-Lipschitzian. Consider the set

$$M = \{(x, y) \in \mathbb{R}^2 | y \leq \sqrt{|x|}\}$$

and, for every integer $k \geq 1$, the set

$$M_k = \{(x, y) \in \mathbb{R}^2 | [y \geq 0 \text{ and } y \leq \sqrt{|x|} - 1/k] \text{ or } [y < 0 \text{ and } y \leq |kx| - 1/k]\}.$$

Then the sequence (M_k) ∂-converges to M (note that $\partial d_M(0) = [-1,1] \times \{0\} = \limsup_{x' \to 0, k \to \infty} \partial d_{M_k}(x')$). But $\limsup_{x' \to 0, k \to \infty} N_{M_k}(x') = \mathbb{R} \times \mathbb{R}_+$ and $N_M(0) = \mathbb{R} \times \{0\}$; hence assertion (nc) is not satisfied. □

The next proposition shows that the concept of ∂-convergence is too strong (for a matter of normalization).

PROPOSITION 2.2. *The set $\mathbb{R}^2 \setminus \mathrm{int}\mathbb{R}^2_+$, i.e., the complementary of the interior of $\mathbb{R}^2_+$ in $\mathbb{R}^2$, cannot be approximated, in the sense of the ∂-convergence, by a sequence of smooth sets.*

Proof of Proposition 2.2. Assume that it is not true, and let (M_k) be a sequence of smooth subsets of $\mathbb{R}^2$ which ∂-converges to $M = \mathbb{R}^2 \setminus \mathrm{int}\mathbb{R}^2_+$. From Proposition 2.1, since M is clearly epi-Lipschitzian, (M_k) converges normally to M; this implies that $(\mathbb{R}^2 \setminus \mathrm{int} M_k)$ converges normally to $\mathbb{R}^2 \setminus \mathrm{int} M = \mathbb{R}^2_+$ (see Proposition 3.1) and we shall prove later (Lemma 4.1) that this implies

$$\perp_{\mathbb{R}^2_+}(0) \subset \limsup_{x \to 0, k \to \infty} N_{\mathbb{R}^2 \setminus \mathrm{int} M_k}(x).$$

Since $v = -(1/\sqrt{2}, 1/\sqrt{2}) \in \perp_{\mathbb{R}^2_+}(0) (= -\mathbb{R}^2_+)$, the above inclusion implies that there is a sequence (x_k) converging to 0, a sequence (v_k) converging to v, and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$, such that, for all k, $x_k \in \mathrm{bd}(\mathbb{R}^2 \setminus \mathrm{int} M_{\varphi(k)}) = \mathrm{bd} M_{\varphi(k)}$ and $v_k \in N_{\mathbb{R}^2 \setminus \mathrm{int} M_{\varphi(k)}}(x_k) = -N_{M_{\varphi(k)}}(x_k)$ (since the set $M_{\varphi(k)}$ is smooth). For k large enough, $v_k \neq 0$; since the set $M_{\varphi(k)}$ is smooth, $-v_k/\|v_k\|$ is the unique element in $N_{M_{\varphi(k)}}(x_k) \cap S$ and hence belongs to $\partial d_{M_{\varphi(k)}}(x_k)$. Then the sequence $(-v_k/\|v_k\|)$ converges to $-v$, which, from assertion (∂c), belongs to $\partial d_M(0)$. Hence,

$$-v = (1/\sqrt{2}, 1/\sqrt{2}) \in \partial d_M(0) = \mathrm{co}\{(1,0), (0,1), (0,0)\},$$

which is a contradiction. □

2.4.2. Involving the limiting normal cone. We now compare the normal convergence with the following concept of $\hat{N}$-convergence, in which one replaces Clarke's normal cone N_M with the limiting normal cone $\hat{N}_M$ (see Mordukhovich [14]) defined previously. More precisely, we say that a sequence (M_k) of closed subsets of $\mathbb{R}^n$ $\hat{N}$-converges to a closed subset $M \subset \mathbb{R}^n$ if it satisfies assertion (sc) together with

$$(\hat{\mathrm{n}}\mathrm{c}) \qquad \limsup \mathrm{G}(\hat{N}_{M_k}) \subset \mathrm{G}(\hat{N}_M).$$

The next proposition shows that the $\hat{N}$-convergence and the ∂-convergence are in fact equivalent.

PROPOSITION 2.3. *Let (M_k) be a sequence of closed subsets of $\mathbb{R}^n$ and let M be a closed subset of $\mathbb{R}^n$. Then (M_k) $\hat{N}$-converges to M if and only if (M_k) ∂-converges to the set M.*

Proof of Proposition 2.3 ($\hat{N}$-convergence $\Rightarrow$ ∂-convergence). Let

$$(x, v) \in \limsup \mathrm{G}(\partial d_{M_k}).$$

Then there is a sequence (x_k) converging to x, a sequence (v_k) converging to v, and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$, such that $x_k \in M_{\varphi(k)}$ and $v_k \in \partial d_{M_{\varphi(k)}}(x_k)$ for all k. Since from (1)

$$\partial d_{M_{\varphi(k)}}(x_k) = \mathrm{co}\Big((\hat{N}_{M_{\varphi(k)}}(x_k) \cap S) \cup \{0\} \Big),$$

from Carathéodory's theorem, there are $n+1$ elements (v_k^i, λ_k^i) $(i \in \{1, \dots, n+1\})$ in

$\mathbb{R}^n \times \mathbb{R}_+$ and an element $\mu_k \in [0,1]$ such that $v_k^i \in \hat{N}_{M_{\varphi(k)}}(x_k) \cap S$, $\sum_{i=1}^{n+1} \lambda_k^i = 1$, and

$$v_k = \mu_k \sum_{i=1}^{m+1} \lambda_k^i v_k^i.$$

Without any loss of generality, we may assume that $(v_k^1, \ldots, v_k^{n+1}, (\lambda_k^1, \ldots, \lambda_k^{n+1}), \mu_k)$ converges to some element $(v^1, \ldots, v^{n+1}, (\lambda^1, \ldots, \lambda^{n+1}), \mu) \in S^{n+1} \times \Sigma \times [0,1]$, where Σ is the unit simplex of $\mathbb{R}^{n+1}$. From assertion ($\hat{\text{n}}$c), we deduce that $v^i \in \hat{N}_M(x) \cap S$ for all $i \in \{1, \ldots, n+1\}$. Then $v \in \text{co}((\hat{N}_M(x) \cap S) \cup \{0\}) = \partial d_M(x)$ from (1). This shows that (M_k) ∂-converges to M.

Proof of Proposition 2.3 (∂-convergence $\Rightarrow$ $\hat{N}$-convergence). Let $(x, p) \in \limsup \mathrm{G}(\hat{N}_{M_k})$. Then there is a sequence (x_k) converging to x, a sequence (p_k) converging to p, and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$, such that $x_k \in M_{\varphi(k)}$ and $p_k \in \hat{N}_{M_{\varphi(k)}}(x_k)$ for all k. If $p = 0$, then clearly $p \in \hat{N}_M(x)$. Assume now that $p \neq 0$. Then for k large enough, $p_k \neq 0$, and $p_k/\|p_k\|$ converges to $p/\|p\|$. Since $p_k/\|p_k\| \in \hat{N}_{M_{\varphi(k)}}(x_k) \cap S \subset \partial d_{M_{\varphi(k)}}(x_k)$ (from (1)), and since (M_k) ∂-converges to M, we deduce that $p/\|p\| \in \partial d_M(x)$. Since

$$\partial d_M(x) \cap S = \text{co}\Big((\hat{N}_M(x) \cap S) \cup \{0\}\Big) \cap S = \hat{N}_M(x) \cap S,$$

we deduce that $p/\|p\| \in \hat{N}_M(x) \cap S$, hence that $p \in \hat{N}_M(x)$. This shows that the sequence (M_k) $\hat{N}$-converges to M. $\square$

Remark 2.10. The proof of the equivalence between the ∂-convergence and the $\hat{N}$-convergence relies heavily on the following equality:

$$\partial d_M(x) = \text{co}\Big((\hat{N}_M(x) \cap S) \cup \{0\}\Big).$$

The difference between the ∂-convergence and the normal convergence might be explained by the fact that the inclusion

$$\partial d_M(x) \subset \text{co}\Big((N_M(x) \cap S) \cup \{0\}\Big)$$

may be strict, even in the epi-Lipschitzian case. Consider $M = \mathbb{R}^2 \setminus \text{int}\mathbb{R}^2_+$.

3. Proof of the approximation result. The first idea to prove Theorem 2.1 is to smooth the distance function d_M by using a classical convolution argument. Indeed, by doing so one directly gets the existence of a normal smooth approximation of M. However, the lipeomorphism properties are more difficult to obtain and our proof will consist of using a more refined argument of convolution (in fact, by using the representation theorem in [6]).

The proof of Theorem 2.1 has three steps. In the first step, we show that (M_k) is an internal (resp., external) smooth approximation of M which satisfies (lip), (lip$^\text{c}$), and (L) if and only if $(\mathbb{R}^n \setminus \text{int}M_k)$ is an external (resp., internal) smooth approximation of $\mathbb{R}^n \setminus \text{int}M$ which satisfies (lip), (lip$^\text{c}$), and (L). In view of this equivalence property, it is sufficient to only show in the following the existence of smooth internal approximations of epi-Lipschitzian sets. In the second step, we improve the representation theorem of Cornet and Czarnecki [6] when the epi-Lipschitzian set is additionally assumed to have a compact boundary. In the third step, the previous representation result allows us to get the approximating sequence. These three steps are proved successively in the following three sections.

3.1. Complementarity property of internal and external approximations.

PROPOSITION 3.1. *Let M and M_k ($k \in \mathbb{N}$) be closed epi-Lipschitzian subsets of $\mathbb{R}^n$. Then the two following assertions are equivalent:*

(i) (M_k) *is a (resp., internal, resp., external, resp., smooth, resp., satisfying* (lip), *resp.,* (lip^c), *resp.,* (L)) *normal approximation of M;*

(ii) $(\mathbb{R}^n \setminus \mathrm{int} M_k)$ *is a (resp., external, resp., internal, resp., smooth, resp., satisfying* (lip^c), *resp.,* (lip), *resp.,* (L)) *normal approximation of* $\mathbb{R}^n \setminus \mathrm{int} M$.

Remark 3.1. Proposition 3.1 is no longer true if we do not assume that M is epi-Lipschitzian. Consider $M = \{0\}$ in $\mathbb{R}$ and $M_k = [-1/k, 1/k]$ for $k \geq 1$. Then (M_k) is a smooth external normal approximation of M and $(\mathbb{R} \setminus \mathrm{int} M_k)$ is not a normal approximation of $\mathbb{R} \setminus \mathrm{int} M$. Consider also the set $M = \{(x,y) \in \mathbb{R}^2 | y \geq \sqrt{|x|}\}$ and the set $M_k = \{(x,y) \in \mathbb{R}^2 | [y \geq 0 \text{ and } y \geq \sqrt{|x|} - 1/k] \text{ or } [y < 0 \text{ and } y \geq (k^3/4)x^2 - (1/4k)]\}$ for $k \geq 1$. Then (M_k) is a smooth internal approximation of M but $(\mathbb{R}^n \setminus \mathrm{int} M_k)$ is not a normal approximation of $\mathbb{R}^n \setminus \mathrm{int} M$. Indeed, $N_{\mathbb{R}^n \setminus \mathrm{int} M}(0) = \mathbb{R} \times \{0\}$; hence the set $\{0\} \times \mathbb{R}_+$, which is contained in $\limsup G(N_{\mathbb{R}^n \setminus \mathrm{int} M_k})$, is not contained in $G(N_{\mathbb{R}^n \setminus \mathrm{int} M})$.

Before proving Proposition 3.1, we prove a claim.

CLAIM 3.1. *Let (M_k) be a sequence of closed subsets of $\mathbb{R}^n$ converging normally to a closed subset $M \subset \mathbb{R}^n$. Then*

$$
\begin{array}{lll}
(\mathrm{int}) & \mathrm{int} M & = \cup_{p \in \mathbb{N}} \mathrm{int}(\cap_{k \geq p} \mathrm{int} M_k); \\
(\mathrm{scc}) & \mathbb{R}^n \setminus \mathrm{int} M & = \limsup(\mathbb{R}^n \setminus \mathrm{int} M_k) = \liminf(\mathbb{R}^n \setminus \mathrm{int} M_k).
\end{array}
$$

Proof of Claim 3.1. We first prove assertion (int).[4] Since

$$\cup_{p \in \mathbb{N}} \cap_{k \geq p} M_k \subset \liminf M_k = M,$$

the inclusion $\cup_{p \in \mathbb{N}} \mathrm{int}(\cap_{k \geq p} \mathrm{int} M_k) \subset \mathrm{int} M$ is immediate. Let us consider

$$x \in \mathrm{int} M \setminus \cup_{p \in \mathbb{N}} \mathrm{int}(\cap_{k \geq p} \mathrm{int} M_k) = \mathrm{int} M \cap [\cap_{p \in \mathbb{N}} \mathrm{cl}(\mathbb{R}^n \setminus \cap_{k \geq p} \mathrm{int} M_k)].$$

Then there is a sequence (x_k) in $\mathbb{R}^n$ converging to x such that, for all $k \in \mathbb{N}$, $x_k \notin \cap_{l \geq k} \mathrm{int} M_l$. Without any loss of generality, we may assume that there is an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that, for all k, $x_k \notin \mathrm{int} M_{\varphi(k)}$. Since $x \in \mathrm{int} M \subset \liminf M_k$, there is a sequence (y_k) in $\mathbb{R}^n$ converging to x such that for all $k \in \mathbb{N}$, $y_k \in M_k$. Since $y_{\varphi(k)} \in M_{\varphi(k)}$, there is $z_k \in \mathrm{bd} M_{\varphi(k)} \cap [x_k, y_{\varphi(k)}]$ and there is $v_k \in N_{M_{\varphi(k)}}(z_k) \cap S$ (see Clarke [4]). The sequence (z_k) converges to x, and we may assume without any loss of generality that v_k converges to some $v \in S$. Hence $(x, v) \in \limsup G(N_{M_k}) \subset G(N_M)$, which implies that $v \in N_M(x)$. The fact that $N_M(x) \neq \{0\}$ contradicts that $x \in \mathrm{int} M$.

Let us now prove assertion (scc). Since $M = \limsup M_k = \cap_{p \in \mathbb{N}} \mathrm{cl}(\cup_{k \geq p} M_k)$, we get that

$$\mathbb{R}^n \setminus M = \cup_{p \in \mathbb{N}} \mathrm{int}(\cap_{k \geq p} \mathbb{R}^n \setminus M_k) \subset \cup_{p \in \mathbb{N}} \cap_{k \geq p} \mathbb{R}^n \setminus \mathrm{int} M_k \subset \liminf(\mathbb{R}^n \setminus \mathrm{int} M_k),$$

hence that $\mathbb{R}^n \setminus \mathrm{int} M \subset \liminf(\mathbb{R}^n \setminus \mathrm{int} M_k) \subset \limsup(\mathbb{R}^n \setminus \mathrm{int} M_k)$. From assertion (int), we get that $\mathbb{R}^n \setminus \mathrm{int} M = \cap_{p \in \mathbb{N}} \mathrm{cl}(\cup_{k \geq p} \mathbb{R}^n \setminus \mathrm{int} M_k) = \limsup(\mathbb{R}^n \setminus \mathrm{int} M_k)$. ∎

Proof of Proposition 3.1. Note that, without any loss of generality, we only need to prove the implication [(i) ⇒ (ii)]. The implication [(ii) ⇒ (i)] can then be deduced

[4] In fact, we prove it in a more general setting later in this paper with a longer proof (Lemma 4.3).

from [(i) $\Rightarrow$ (ii)], applying the result to the set $\mathbb{R}^n \setminus \mathrm{int}M$, since $\mathbb{R}^n \setminus \mathrm{int}M$ is epi-Lipschitzian, since $\mathbb{R}^n \setminus \mathrm{int}(\mathbb{R}^n \setminus \mathrm{int}M) = \overline{\mathrm{int}M} = M$[5] and $\mathbb{R}^n \setminus \mathrm{int}(\mathbb{R}^n \setminus \mathrm{int}M_k) = \overline{\mathrm{int}M_k} = M_k$ for all k.

In view of Claim 3.1, the sequence $(\mathbb{R}^n \setminus \mathrm{int}M_k)$ and the set $\mathbb{R}^n \setminus \mathrm{int}M$ clearly satisfy assertion (sc) of Definition 2.1. Let us prove that they satisfy assertion (nc). Let $(x,p) \in \limsup G(N_{\mathbb{R}^n \setminus \mathrm{int}M_k})$. Without any loss of generality, we may assume that $p \neq 0$, hence that $x \in \limsup \mathrm{bd}(\mathbb{R}^n \setminus \mathrm{int}M_k) = \limsup \mathrm{bd}M_k$, since $\mathrm{bd}(\mathbb{R}^n \setminus \mathrm{int}M_k) = (\mathbb{R}^n \setminus \mathrm{int}M_k) \setminus \mathrm{int}(\mathbb{R}^n \setminus \mathrm{int}M_k) = \overline{\mathrm{int}M_k} \setminus \mathrm{int}M_k = \mathrm{bd}M_k$ for all k (since M_k is epi-Lipschitzian). This implies that $x \in M \cap (\mathbb{R}^n \setminus \mathrm{int}M) = \mathrm{bd}M = \mathrm{bd}(\mathbb{R}^n \setminus \mathrm{int}M)$. Then, since M_k is epi-Lipschitzian, $N_{\mathbb{R}^n \setminus \mathrm{int}M_k}(x') = -N_{M_k}(x')$ for all $x' \in \mathrm{bd}M_k$, hence $(x,-p) \in \limsup G(N_{M_k})$, hence $(x,-p) \in G(N_M)$, which implies that $(x,p) \in G(N_{\mathbb{R}^n \setminus \mathrm{int}M})$, since $N_{\mathbb{R}^n \setminus \mathrm{int}M}(x) = -N_M(x)$. We proved that $(\mathbb{R}^n \setminus \mathrm{int}M_k)$ is an approximation of $\mathbb{R}^n \setminus \mathrm{int}M$. If (M_k) is a smooth (resp., internal, resp., external, resp., satisfying (lip), resp., (lip^c), resp., (L)) approximation of M, then $(\mathbb{R}^n \setminus \mathrm{int}M_k)$ is clearly a smooth (resp., external, resp., internal, resp., satisfying (lip), resp., (lip^c), resp., (L)) approximation of $\mathbb{R}^n \setminus \mathrm{int}M$. $\square$

3.2. A representation theorem. We first state a representation theorem of M when $\mathrm{bd}M$ is compact.

THEOREM 3.1. *Let M be a closed epi-Lipschitzian subset of $\mathbb{R}^n$ with compact boundary $\mathrm{bd}M$. Then there is a function $f_M : \mathbb{R}^n \to \mathbb{R}$ which is a quasi-smooth inequality representation of M in the following sense:*

(i) *f_M is locally Lipschitzian on $\mathbb{R}^n$ and C^∞ on $\mathbb{R}^n \setminus \mathrm{bd}M$;*

(ii) *$M = \{x \in \mathbb{R}^n | f_M(x) \leq 0\}$;*

(iii) *$\mathrm{bd}M = \{x \in \mathbb{R}^n | f_M(x) = 0\}$;*[6]

(iv) *$0 \notin \partial f_M(x)$ if $f_M(x) = 0$;*

(v) *$N_M(x) \cap -N_{\mathbb{R}^n \setminus \mathrm{int}M}(x) = \cup_{\lambda \geq 0} \lambda \partial f_M(x)$ for all $x \in \mathrm{bd}M$.*

Furthermore, one can assume that for some $\varepsilon > 0$:

(vi) *$f_M^{-1}([-\varepsilon,\varepsilon])$ is compact;*

(vii) *$\forall x \in f_M^{-1}([-\varepsilon,\varepsilon])$, $\mathrm{co}\partial f_M(\overline{B}(x,\varepsilon)) \cap \overline{B}(0,\varepsilon) = \emptyset$.*

Proof of Theorem 3.1. The existence of a function f satisfying assertions (i)–(v) is exactly Theorem 2.1 of Cornet and Czarnecki [6] (in which the closed set M is not assumed to have a compact boundary).

Let f be a quasi-smooth representation of M on $\mathbb{R}^n$ (i.e., satisfying assertions (i)–(v)). Let $\alpha : \mathbb{R}^n \to [0,1]$ be a C^∞ function such that $\alpha(x) = 0$ if $x \in \overline{B}(\mathrm{bd}M, 1/2)$ and $\alpha(x) = 1$ if $x \notin B(\mathrm{bd}M, 1)$. We define the function $f_M : \mathbb{R}^n \to \mathbb{R}$ for all $x \in \mathbb{R}^n$ by

$$\begin{array}{lll} f_M(x) = (1-\alpha(x))f(x) + \alpha(x)\mathrm{sgn} f(x) & \text{if} & f(x) \neq 0, \\ f_M(x) = 0 & \text{if} & f(x) = 0, \end{array}$$

denoting $\mathrm{sgn}\, t = t/|t|$ if $t \in \mathbb{R} \setminus \{0\}$.

Proof of (i)–(iii). The function f_M clearly satisfies assertions (i), (ii), and (iii) of Theorem 3.1.

Proof of (iv) *and* (v). Let $x \in \mathrm{bd}M$. Since $\alpha = 0$ on a neighborhood of x, one gets that $\partial f_M(x) = \partial f(x)$; hence assertions (iv) and (v) of Theorem 3.1 are satisfied.

[5]This is a classical result on epi-Lipschitzian sets (see, for example, Cornet and Czarnecki [6]).

[6]This assertion is a consequence of assertions (ii) and (iv).

Proof of (vi). Since $f_M^{-1}((-1,1)) \subset B(\mathrm{bd}M, 1)$, then $\mathrm{cl}(f_M^{-1}((-1,1)))$ is compact.

Proof of (vii). It is a consequence of the following lemma (taking $m = n$, $K = \mathrm{bd}M = f_M^{-1}(\{0\})$, and $\Phi = \partial f_M$) and of the fact that $f_M^{-1}([-\varepsilon, \varepsilon]) \subset B(\mathrm{bd}M, r)$ for some $\varepsilon \in (0, 1]$ (since $B(\mathrm{bd}M, r)$ is an open set containing the intersection of compact sets $\cap_{\varepsilon \in (0,1]} f_M^{-1}([-\varepsilon, \varepsilon])$).

LEMMA 3.1. *Let K be a compact subset of $\mathbb{R}^n$ and let Φ be a u.s.c. correspondence from $\mathbb{R}^n$ to $\mathbb{R}^m$, with nonempty compact convex values, such that $0 \notin \Phi(x)$ for every $x \in K$. Then there exists $r > 0$ such that*

$$\mathrm{co}\Phi(\overline{B}(x, r)) \cap \overline{B}(0, r) = \emptyset, \textit{ for all } x \in B(K, r).$$

Proof of Lemma 3.1 *(by contraposition).* Suppose that there exists a sequence (x^k) in $\mathbb{R}^n$ such that, for all k, $x^k \in B(K, 1/k)$ and

$$\mathrm{co}\Phi(\overline{B}(x^k, 1/k)) \cap \overline{B}(0, 1/k) = \emptyset.$$

From Carathéodory's theorem, there exist $n+1$ elements $(x_i^k, y_i^k, \lambda_i^k)$ $(i = 1, \ldots, n+1)$ in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+$ such that $x_i^k \in \overline{B}(x^k, 1/k)$, $y_i^k \in \Phi(x_i^k)$, $\sum_{i=1}^{m+1} \lambda_i^k = 1$, and

$$\left\| \sum_{i=1}^{m+1} \lambda_i^k y_i^k \right\| \leq 1/k.$$

Without loss of generality, we assume that the sequence $(x^k, \lambda_1^k, \ldots, \lambda_{m+1}^k, y_1^k, \ldots, y_{m+1}^k)$ converges to some element $(x^*, \lambda_1^*, \ldots, \lambda_{m+1}^{k*}, y_1^*, \ldots, y_{m+1}^*) \in K \times \Sigma \times \mathbb{R}^{m(m+1)}$, since the sequence belongs to the compact set $\overline{B}(K, 1) \times \Sigma \times \Phi(\overline{B}(K, 1))^{m+1}$, where Σ is the unit simplex of $\mathbb{R}^{m+1}$ and the set $\Phi(\overline{B}(K, 1))$ is clearly bounded (since $\Phi(\overline{B}(K, 1))$ is the image of the compact set $\overline{B}(K, 1)$ by the u.s.c. correspondence Φ). However, for all $i \in \{1, \ldots, m+1\}$, the sequence (x_i^k) also converges to x^* (since from above $\|x_i^k - x^k\| \leq 1/k$).

Taking the limit when $k \to \infty$, we get $0 = \sum_{i=1}^{m+1} \lambda_i^* y_i^*$ and $y_i^* \in \Phi(x^*)$ for all $i \in \{1, \ldots, m+1\}$, since the correspondence Φ is u.s.c. Consequently, $0 \in \Phi(x^*)$ since $\Phi(x^*)$ is convex. Since $x^* \in K$, this contradicts the assumption $0 \notin \Phi(x^*)$. ∎

Remark 3.2. When $\mathrm{bd}M$ is not compact, assertion (vi) is clearly false for any quasi-smooth inequality representation of M. The following example shows that assertion (vii) may not be true if $\mathrm{bd}M$ is not assumed to be compact. Consider the set $M = \{(x, y) \in \mathbb{R}^2 | (x \leq 0) \text{ or } (x > 0 \text{ and } y \in [-1/x, 1/x])\}$.

3.3. Proof of Theorem 2.1. In view of Proposition 3.1, we only need to show the existence of smooth internal approximations of epi-Lipschitzian sets. Let f_M be a quasi-smooth representation of M satisfying the conclusions of Theorem 3.1 for some $\varepsilon > 0$; we let

$$M_k = \{x \in \mathbb{R}^n | f_M(x) \leq -\varepsilon/k\}$$

for every integer $k \geq 1$. We shall prove that (M_k) is a smooth internal approximation of M satisfying the conclusions of Theorem 2.1.

It is clearly an increasing sequence, and the set-convergence assertion (sc) is immediate. The sets M_k are clearly smooth, since f_M is C^∞ on $\{x \in \mathbb{R}^n | f_M(x) \neq 0\}$ and since $\nabla f_M(x) \neq 0$ when $x \in \mathrm{bd}M_k$ (in that case, $f_M(x) = -\varepsilon/k$, $f_M(x) \in [-\varepsilon, \varepsilon]$ and $0 \notin \partial f_M(x) = \{\nabla f_M(x)\}$ from Theorem 3.1, assertion (vii)).

Proof of the normal convergence assertion (nc). Let $(x,v) \in \limsup G(N_{M_k})$. Then there exist two sequences (x_k) and (v_k) in $\mathbb{R}^n$ and an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, such that (x_k) converges to x, (v_k) converges to v, and, for all k, $x_k \in M_{\varphi(k)}$ and $v_k \in N_{M_{\varphi(k)}}(x_k)$. Then, for all k, there is $\lambda_k \geq 0$ such that $v_k = \lambda_k \nabla f_M(x_k)$. Since f_M is Lipschitzian on a neighborhood of x, the sequence $(\nabla f_M(x_k))$ is bounded. Without any loss of generality, we may assume that it converges to some $u \in \mathbb{R}^n$. Since the correspondence ∂f_M is u.s.c., $u \in \partial f_M(x)$, hence $u \neq 0$ since $0 \notin \partial f_M(x)$. This implies that the sequence (λ_k) converges to $\lambda = \|v\|/\|u\|$ and $v = \lambda u$ with $\lambda \geq 0$ and $u \in \partial f_M(x)$. Hence $v \in N_M(x)$.

Proof of the lipeomorphism assertions (lip) *and* (lip$^{\mathrm{c}}$). Let us now prove that the sets M and M_k are lipeomorphic for all k. In view of Bonnisseau–Cornet [2, Theorem 2.5],[7] assertion (lip) is a clear consequence of the following facts:

$$\begin{array}{l} M_k = \{x \in \mathbb{R}^n | f_M(x) \leq -\varepsilon/k\} \text{ for every } k, \\ M = \{x \in \mathbb{R}^n | f_M(x) \leq 0\}, \\ f_M^{-1}([-\varepsilon/k, 0]) \text{ is compact}, \\ 0 \notin \partial f_M(x) \text{ when } f_M(x) \in [-\varepsilon/k, 0]. \end{array}$$

The proof that the sets $\mathbb{R}^n \setminus \mathrm{int} M_k$ and $\mathbb{R}^n \setminus \mathrm{int} M$ are lipeomorphic is a consequence of the same result, since $\mathbb{R}^n \setminus \mathrm{int} M_k = \{x \in \mathbb{R}^n | f_M(x) \geq -\varepsilon/k\}$ and $\mathbb{R}^n \setminus \mathrm{int} M = \{x \in \mathbb{R}^n | f_M(x) \leq 0\}$.

Proof of (L). In view of assertion (sc) of Definition 2.1, since the sequence (M_k) is increasing, and since $\mathrm{bd} M$ is compact, there clearly is an integer k_0 such that $\mathrm{bd} M_k \subset B(\mathrm{bd} M, \varepsilon)$ for all $k \geq k_0$. Let us consider $k \geq k_0$, and let L be the Lipschitz constant of f_M on $B(\mathrm{bd} M, \varepsilon)$. We first prove that

$$\varepsilon[(1/k) - 1/(k+1)] \leq L \min\{\|x - y\|, x \in \mathrm{bd} M_k, y \in \mathrm{bd} M_{k+1}\}.$$

Indeed, if $x \in \mathrm{bd} M_k$ and $y \in \mathrm{bd} M_{k+1}$, then $f_M(x) = -\varepsilon/k$ and $f_M(y) = -\varepsilon/(k+1)$. Then $\varepsilon[(1/k) - 1/(k+1)] = |f_M(x) - f_M(y)| \leq L\|x - y\|$. We now end the proof of Theorem 2.1 by proving that

$$\delta(\mathrm{bd} M_k, \mathrm{bd} M_{k+1}) \leq [(1/k) - 1/(k+1)].$$

We first consider $x \in \mathrm{bd} M_k$. Since f_M satisfies Theorem 3.1, assertion (vii), we can separate the compact convex sets $\mathrm{co}\partial f_M(\overline{B}(x,\varepsilon))$ and $\overline{B}(0,\varepsilon)$. Hence there is $p \in S(0,1)$ such that $(p|y) > \varepsilon$ for all $y \in \mathrm{co}\partial f_M(\overline{B}(x,\varepsilon))$. Then, considering the map $t \mapsto f_M(x + tp)$, from the mean-value theorem (see Clarke [4]) one easily proves that $f_M(x+tp) - f_M(x) \geq \varepsilon t$ for all $t \in [0,\varepsilon]$. Then $f_M(x + \varepsilon p) \geq -\varepsilon/k + \varepsilon^2 > 0$ (if k is large enough), hence there is $t \in [0,\varepsilon]$ such that $f_M(x+tp) = -\varepsilon/(k+1)$. Hence $\varepsilon t \leq f_M(x+tp) - f_M(x) = \varepsilon[(1/k) - 1/(k+1)]$. This implies that $d(x, \mathrm{bd} M_{k+1}) \leq \|tp\| \leq [(1/k) - 1/(k+1)]$. One proves that $d(x, \mathrm{bd} M_k) \leq [(1/k) - 1/(k+1)]$ for all $x \in \mathrm{bd} M_{k+1}$ in the same way. Hence $\delta(\mathrm{bd} M_k, \mathrm{bd} M_{k+1}) \leq [(1/k) - 1/(k+1)]$. □

Remark 3.3. From the above proof, we note that assertion (L) is satisfied by the sequence $M_k = \{x \in \mathbb{R}^n | f_M(x) \leq \varepsilon_k\}$, where $f_M : \mathbb{R}^n \to \mathbb{R}$ is a quasi-smooth representation of M and (ε_k) is a strictly decreasing sequence of positive real numbers converging to zero. However, there may exist normal converging sequences which cannot be represented as above. Consider the example in Remark 2.5.

[7] Bonnisseau and Cornet [2] prove a homeomorphism result, but only a slight change in their proof gives a lipeomorphism result.

4. Proof of the lipeomorphism result.

In view of Proposition 3.1, the proof of Theorem 2.3 has three steps. We first show that the sets M and M_k are epi-Lipschitzian for k large enough. In the second step, we show the existence of a Lipschitzian transverse field between M and M_k, in a sense that will be explained later. In the third step of the proof we show that, if there is a such transverse field between two sets M and N, then M and N are (epi-Lipschitzian and) lipeomorphic. These three steps are proved successively in the following three sections.

4.1. The sets M and M_k are epi-Lipschitzian.

PROPOSITION 4.1. *Let (M_k) be a sequence of closed subsets of $\mathbb{R}^n$ such that* $\mathrm{bd} M_k \subset K$ *for all k, for some fixed compact subset $K \subset \mathbb{R}^n$, and such that*

(**) $\mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ *is pointed for all* $x \in \limsup M_k$.

Then the set $M = \limsup M_k$ and the set M_k, for k large enough, are epi-Lipschitzian.

Proof of Proposition 4.1. We first prove that M_k is epi-Lipschitzian for k large enough. Assume that it is not true. Then there is a sequence (x_k) in $\mathbb{R}^n$, a sequence (p_k) in S, and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$, such that, for all $k \in \mathbb{N}$, $x_k \in M_{\varphi(k)}$, $p_k \in N_{M_{\varphi(k)}}(x_k)$, and $-p_k \in N_{M_{\varphi(k)}}(x_k)$. Then $x_k \in \mathrm{bd} M_{\varphi(k)} \subset K$; hence we may assume without any loss of generality that the sequence x_k converges to some $x \in K$ and that the sequence p_k converges to some $p \in S$. Then $x \in M$ (since $M = \limsup M_k$) and p and $-p$ belong to $\limsup_{x' \to x, k \to \infty} N_{M_k}(x')$, which contradicts the fact that $\mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ is pointed for every $x \in M$.

We now prove that M is epi-Lipschitzian. Let $x \in M$. Then, since the set $\mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ is pointed, it is sufficient to prove that $N_M(x) \subset \mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ for all $x \in M$, which we do in the next lemma, which generalizes Lemma 6.2 from Benoist [1] (see also Kruger and Mordukhovich [11, Theorem P.3] and Ioffe [10, Theorem 3]).

LEMMA 4.1. *Let (M_k) be a sequence of closed subsets of $\mathbb{R}^n$, and let* $M = \limsup M_k$. *Then, for all $x \in M$*

(i) $\perp_M(x) \subset \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$;[8]

(ii) $N_M(x) \subset \mathrm{cl}\ (\mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x'))$;

(iii) *if we additionally assume that the set* $\mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ *is pointed, then we can suppress* cl *in the above assertion, i.e., formally*

$$N_M(x) \subset \mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x').$$

Proof of Lemma 4.1. Proof of (i). Let $\overline{x} \in M$ and $\overline{p} \in \perp_M(\overline{x}) \setminus \{0\}$. One easily notices from the definition of $\perp_M(\overline{x})$ that, for $\mu > 0$ small enough,

$$\overline{B}(\overline{x} + \mu\overline{p}, \mu\|\overline{p}\|) \cap M = \{\overline{x}\}. \tag{2}$$

We define the function $\psi : \mathbb{R}^n \to \mathbb{R}$ by $\psi(x) = (1/2)\|x - (\overline{x} + \mu\overline{p})\|^2$. Then, for all integer k there is a solution $x_k \in \mathbb{R}^n$ of the following minimization problem:

$$(P_k) \begin{cases} \text{minimize} \quad \psi(x), \\ x \in M_k. \end{cases}$$

Then, from (2), $\overline{x} + \mu\overline{p} \notin M$. Hence, since $M = \limsup M_k$, $\overline{x} + \mu\overline{p} \notin M_k$ for k

[8] One can easily replace assertion (i) with $\perp_M(x) \subset \limsup_{x' \to x, k \to \infty} \perp_{M_k}(x')$ or, equivalently, $G(\perp_M) \subset \limsup G(\perp_{M_k})$. However, without the convexification of the right-hand side, assertion (iii) does not hold in general.

large enough; hence $x_k \in \mathrm{bd} M_k$. Then x_k satisfies the following first-order necessary condition associated with (P_k) (see Clarke [4]):

$$-x_k + \overline{x} + \mu\overline{p} = -\nabla\psi(x_k) \in N_{M_k}(x_k). \tag{3}$$

Let us show that the sequence (x_k) admits a bounded subsequence. Since $\overline{x} \in M = \limsup M_k$, there is a sequence $(\overline{x}_k)$ converging to $\overline{x}$ and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$, such that $\overline{x}_k \in M_{\varphi(k)}$ for all k. Then, for every k, $\overline{x}_k$ satisfies the constraint of $(P_{\varphi(k)})$; hence

$$\psi(x_{\varphi(k)}) = (1/2)\|x_{\varphi(k)} - \overline{x} - \mu\overline{p}\|^2 \leq \psi(\overline{x}_k) = (1/2)\|\overline{x}_k - \overline{x} - \mu\overline{p}\|^2. \tag{4}$$

Because the sequence $(\overline{x}_k)$ is convergent, hence bounded, this implies that $(x_{\varphi(k)})$ is bounded. Without any loss of generality, we may assume that the sequence $(x_{\varphi(k)})$ converges to some $x \in \mathbb{R}^n$. Since $x_{\varphi(k)} \in M_{\varphi(k)}$ for all $k \in \mathbb{N}$, we get that $x \in M$. Since the sequence $(\overline{x}_k)$ converges to $\overline{x}$, $\psi(\overline{x}_k)$ converges to $(1/2)\mu^2\|\overline{p}\|^2$, hence (4) implies that $\psi(x) \leq (1/2)\mu^2\|\overline{p}\|^2$ and hence that $x \in \overline{B}(\overline{x} + \mu\overline{p}, \mu\|\overline{p}\|)$. In view of (2), since additionally $x \in M$, we get that $x = \overline{x}$. Letting $p_{\varphi(k)} = (1/\mu)(-x_{\varphi(k)} + \overline{x} + \mu\overline{p})$, we proved that $\overline{p} = \lim_{k\to\infty} p_{\varphi(k)}$ with $p_{\varphi(k)} \in N_{M_{\varphi(k)}}(x_{\varphi(k)})$.

Proof of (ii) *and* (iii). Since $A = \limsup_{x'\to\overline{x},k\to\infty} N_{M_k}(x')$ is a cone, since the correspondence $x \mapsto \limsup_{x'\to x,k\to\infty} N_{M_k}(x')$ is closed at $\overline{x}$, we get that $N_M(\overline{x}) \subset \mathrm{cl}(\mathrm{co}A)$, which proves (ii). If we additionally assume that $\mathrm{co}A$ is pointed, then $\mathrm{cl}(\mathrm{co}A) = \mathrm{co}A$, since A is closed (recalling that $\mathrm{co}A$ is closed, when A is a closed cone such that $\mathrm{co}A$ is pointed). $\square$

4.2. A transverse field between M and M_k.

PROPOSITION 4.2. *Let M and (M_k) satisfy the hypothesis of Theorem 2.3. Then, for k large enough, there exists a Lipschitzian transverse field between the two sets M and $N = M_k$ in the following sense:*

$$(\mathrm{T}) \begin{cases} \text{There is bounded Lipschitzian map } F : \mathbb{R}^n \to \mathbb{R}^n, \text{ such that} \\ \text{(a)} \quad M \setminus \mathrm{int}N \text{ and } N \setminus \mathrm{int}M \text{ are compact;} \\ \text{(b)} \quad \forall x \in \mathrm{bd}M \ (\textit{resp.}, \mathrm{bd}N),\ F(x) \in \mathrm{int}T_M(x),\ (\textit{resp.}, F(x) \in \mathrm{int}T_N(x)); \\ \text{(c)} \quad \text{for all } x \text{ in an open neighborhood } U \text{ of } M \setminus \mathrm{int}N \cup N \setminus \mathrm{int}M, \\ \qquad \exists (t,t') \in \mathbb{R}^2, \varphi(t,x) \in \mathrm{bd}M, \varphi(t',x) \in \mathrm{bd}N.^{9} \end{cases}$$

Remark 4.1. From [5], the assertion $\forall x \in \mathrm{bd}M$, $F(x) \in \mathrm{int}T_M(x)$ is equivalent to the weaker one, $\forall x \in \mathrm{bd}M$, $F(x) \in \mathrm{int}\mathcal{T}_M(x)$, where $\mathcal{T}_M(x)$ is Bouligand's tangent cone to M at x.

Note that from Proposition 4.1 the sets M and M_k are epi-Lipschitzian for k large enough. Before proving Proposition 4.2, we prove some preliminary lemmas.

4.2.1. Preliminary lemmas.

The first lemma gives an extended Gauss correspondence, an essential step to the construction of the transverse field.

LEMMA 4.2. *There is $r > 0$ and a correspondence G from $\mathbb{R}^n$ to $\mathbb{R}^n$ which is u.s.c., with nonempty compact convex values and such that*

(i) $G_M(x) = N_M(x) \cap S \subset G(x)$ *for all* $x \in \mathrm{bd}M$*;*

(ii) $G_{M_k}(x) = N_{M_k}(x) \cap S \subset G(x)$ *for all* $x \in \mathrm{bd}M_k$*, for k large enough;*

[9] Where φ is the flow of the following differential equation:

$$(E)\ \dot{x}(t) = F(x(t)),\ x(0) = x,$$

i.e., $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the unique C^1 map such that $t \mapsto \varphi(t,x)$ is the (unique) maximal solution of (E) (and is defined on $\mathbb{R}$ since F is bounded).

(iii) $0 \notin G(x)$ *for all* $x \in B(\limsup \mathrm{bd} M_k, r)$,
recalling that S *is the unit sphere of* $\mathbb{R}^n$.

Proof of Lemma 4.2. We let

$$G_\infty(x) = \mathrm{co}\Big(\mathrm{co}\Big(\limsup_{x' \to x, k \to \infty} N_{M_k}(x')\Big) \cap S\Big) \text{ for all } x \in \limsup \mathrm{bd} M_k.$$

CLAIM 4.1. *The correspondence* G_∞*, from* $\limsup \mathrm{bd} M_k$ *to* $\mathbb{R}^n$*, is u.s.c. with nonempty compact convex values and* $0 \notin G_\infty(x)$ *for every* $x \in \limsup \mathrm{bd} M_k$.

Proof of Claim 4.1. The correspondence $x \mapsto \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ has a closed graph, hence the correspondence $x \mapsto \limsup_{x' \to x, k \to \infty} N_{M_k}(x') \cap S$ is u.s.c., with compact values. We now show that it has nonempty values. Let $x \in \limsup \mathrm{bd} M_k$. Then there is a sequence (x_k) in $\mathbb{R}^n$ converging to x and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that $x_k \in \mathrm{bd} M_{\varphi(k)}$ for all k. Then there is a sequence (p_k) in S such that $p_k \in N_{M_{\varphi(k)}}(x_k)$ for all k (see Clarke [4]). Let p be a cluster point of the sequence (p_k). Then $p \in \limsup_{x' \to x, k \to \infty} N_{M_k}(x') \cap S \subset G_\infty(x)$. We proved that the correspondence G_∞ is u.s.c., with nonempty compact convex values. Finally $0 \notin G_\infty(x)$, for all $x \in \limsup \mathrm{bd} M_k \subset M$, since $\mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ is pointed. ☐

In view of the extension theorem of Cellina [3],[10] we let $\widehat{G_\infty}$ be an extension of G_∞ on $\mathbb{R}^n$, which is u.s.c., with nonempty compact convex values. For $r > 0$ we define the correspondence G_r from $\mathbb{R}^n$ to $\mathbb{R}^n$ by

$$G_r(x) = \mathrm{co}\, \overline{B}\Big(\widehat{G_\infty}(\overline{B}(x, r)), r\Big).$$

It has clearly nonempty convex values. Also, the correspondence G_r has compact values (since it is the sum of a compact set and of the convex hull of the image of a compact set by a u.s.c. correspondence). The correspondence G_r is clearly u.s.c. (recalling that, if Φ is a u.s.c. correspondence with convex values from $\mathbb{R}^n$ to $\mathbb{R}^n$ and if $r > 0$, then the correspondence $\mathrm{co}\Phi$, and the correspondences Φ_1 and Φ_2, defined by $\Phi_1(x) = \Phi(\overline{B}(x, r))$ and $\Phi_2(x) = \overline{B}(\Phi(x), r)$, respectively, are also u.s.c.).

Proof of (i). Since $M = \limsup M_k$ and since $\mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ is pointed, then $N_M(x) \subset \mathrm{co} \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$ for all $x \in \mathrm{bd} M$ (Lemma 4.1). Since $M = \limsup M_k$, then, from Lemma 4.1, $\mathrm{bd} M \subset \limsup \mathrm{bd} M_k$ (this can also be proved more directly). Then assertion (i) is a direct consequence of the definition of G_r.

Proof of (ii). We now prove that, for k large enough, $N_{M_k}(x) \cap S \subset G_r(x)$ for all $x \in \mathrm{bd} M_k$. Suppose that it is not true, then we may assume without any loss of generality that there are two sequences (x_k) and (p_k) in $\mathbb{R}^n$ such that, for all k, $x_k \in \mathrm{bd} M_k$, $p_k \in N_{M_k}(x) \cap S$, and $p_k \notin G_r(x_k)$. Since the sequence (x_k, p_k) belongs to the compact set $K \times S$, without any loss of generality, we may assume that it converges to an element $(x, p) \in K \times S$. Then $x \in M$ (since $M = \limsup M_k$) and $p \in \limsup_{x' \to x, k \to \infty} N_{M_k}(x')$. For k large enough, $x_k \in \overline{B}(x, r)$ and $p_k \in \overline{B}(p, r)$, hence $p_k \in \widehat{G_\infty}(\overline{B}(x_k, r)) + \overline{B}(0, r) \subset G_r(x_k)$, which is a contradiction. ☐

Proof of (iii). Since $\limsup \mathrm{bd} M_k \subset K$, it is compact. The end of the proof consists of choosing $r > 0$ as in Lemma 3.1 (taking $m = n$, considering the compact

[10] Let Φ be a u.s.c. correspondence, with nonempty compact convex values defined on a closed set $X \subset \mathbb{R}^n$, with values in $\mathbb{R}^m$. Then there is a u.s.c. correspondence $\hat{\Phi}$, with nonempty compact convex values, defined on $\mathbb{R}^n$ with values in $\mathbb{R}^m$, such that $\hat{\Phi}|_X = \Phi$ and such that $\hat{\Phi}(\mathbb{R}^n) \subset \mathrm{co}\Phi(X)$.

set $\limsup \mathrm{bd} M_k$ and the correspondence G_∞), recalling that $0 \notin G_\infty(x)$ if $x \in \limsup \mathrm{bd} M_k$ (Claim 4.1). □

The second lemma is a consequence of Lemma 4.2 and eliminates the situation where $M = \overline{B}(0,1)$ and $M_k = M \setminus B(0,1/k)$ for $k \geq 1$ (see Remark 2.3).

LEMMA 4.3.

$$\text{(int)} \qquad \mathrm{int} M = \cup_{p\in\mathbb{N}} \mathrm{int}(\cap_{k\geq p} \mathrm{int} M_k).$$

Proof of Lemma 4.3. We recall that the inclusion $\cup_{p\in\mathbb{N}} \mathrm{int}(\cap_{k\geq p} \mathrm{int} M_k) \subset \mathrm{int} M$ is an immediate consequence of the equality $\liminf M_k = M$. We now prove the converse inclusion. Let $x \in \mathrm{int} M$. There is $\varepsilon > 0$ such that $B(x,\varepsilon) \subset M$, and from Lemma 3.1, such that $0 \notin \mathrm{co} G(B(x,\varepsilon))$. Assume that $x \notin \cup_{p\in\mathbb{N}} \mathrm{int}(\cap_{k\geq p} \mathrm{int} M_k)$. Then there is a sequence (x_k) converging to x and a subsequence $(M_{\varphi(k)})$ such that, for all k, $x_k \in \mathrm{bd} M_{\varphi(k)}$ (see the proof of Claim 3.1). Hence $x \in \limsup \mathrm{bd} M_{\varphi(k)} \subset \limsup \mathrm{bd} M_k$. From a classical separation argument, there is $p \in S$ and a real number $a > 0$ such that $(p|y) > a$ for all $y \in \mathrm{co} G(B(x,\varepsilon))$. We may assume without any loss of generality that $a = \varepsilon$. Then, since $\mathrm{co} G(\overline{B}(x,\varepsilon))$ is bounded (it is the convex hull of the image of a compact set by a u.s.c. correspondence), there is $\varepsilon' > 0$ such that $(p'|y) > 0$ for all $p' \in B(p,\varepsilon')$ and for all $y \in \mathrm{co} G(B(x,\varepsilon))$. Without any loss of generality, we may assume that $\varepsilon' = \varepsilon$. Then $-p' \in \mathrm{int} T_{M_k}(x')$ for all $x' \in B(x,\varepsilon) \cap \mathrm{bd} M_k$ and for all $p' \in B(p,\varepsilon')$, if k is large enough since, from Lemma 4.2, $N_{M_k}(x') \cap S \subset G(x')$.

Then the following claim implies that, for k large enough, $x_k + tp' \notin M_{\varphi(k)}$ for all $p' \in B(p,\varepsilon/2)$ and all $t \in (0,\varepsilon/2)$, hence that $B(x + (\varepsilon/3)p, (\varepsilon^2/6)) \cap M_{\varphi(k)} = \emptyset$, hence that $x + (\varepsilon/3)p \notin \limsup M_{\varphi(k)}$, contradicting the fact that $M = \limsup M_{\varphi(k)}$.[11] □

CLAIM 4.2. *Let M be a closed epi-Lipschitzian subset of $\mathbb{R}^n$, let $x \notin \mathrm{int} M$, $\varepsilon > 0$, and $p \in S$ such that $-p \in \mathrm{int} T_M(x')$ for all $x' \in B(x,\varepsilon) \cap M$. Then $x + tp \notin M$ for $t \in (0,\varepsilon)$.*

Proof of Claim 4.2. Assume that $x + tp \in M$ for some $t \in [0,\varepsilon)$. Since $x \notin \mathrm{int} M$, we may assume without any loss of generality that $x + tp \in \mathrm{bd} M$. Then $x + tp \in B(x,\varepsilon)$, hence $-p \in \mathrm{int} T_M(x + tp)$. We recall that from Rockafellar [16]

$$\begin{aligned}\mathrm{int} T_M(x+tp) = \{\, v \in \mathbb{R}^n | \exists \alpha > 0, y + \lambda w \in M \\ \text{for all } (y,w,\lambda) \in (B(x+tp,\alpha) \cap M) \times B(w,\alpha) \times [0,\alpha)\}.\end{aligned}$$

Let $\alpha > 0$ be chosen as above. Then $(x+tp) - (\alpha/2)p \in M$. Hence $(x+tp) - (\alpha/2)p + (\alpha/2)p' \in M$ for all $p' \in B(p,\alpha)$; hence $x + tp \in \mathrm{int} M$, which is a contradiction. □

4.2.2. Proof of Proposition 4.2. We now prove that, for k large enough, there is a Lipschitzian transverse field between the sets M and M_k. We recall that $\mathrm{bd} M \subset \limsup \mathrm{bd} M_k \subset K$, hence that it is compact. We let $U = B(\mathrm{bd} M, \varepsilon)$ for a given real number $\varepsilon > 0$.

Proof of (a). Since $B(M,\varepsilon) = M \cup B(\mathrm{bd} M,\varepsilon)$ and since $B(\mathbb{R}^n \setminus \mathrm{int} M,\varepsilon) = (\mathbb{R}^n \setminus \mathrm{int} M) \cup B(\mathrm{bd} M,\varepsilon)$, the following claim clearly implies that $M_k \setminus \mathrm{int} M \subset B(\mathrm{bd} M,\varepsilon) = U$ and that $M \setminus \mathrm{int} M_k \subset U$, hence that $M_k \setminus \mathrm{int} M$ and $M \setminus \mathrm{int} M_k$ are compact, since $\overline{B}(\mathrm{bd} M,\varepsilon)$ is clearly compact.

[11] Recalling that the equality $M = \limsup M_k = \liminf M_k$ implies that $M = \limsup M_{\varphi(k)} = \liminf M_{\varphi(k)}$.

CLAIM 4.3. *Let ε be a positive real number. Then, for k large enough,*[12]

$$\begin{array}{lll} M_k & \subset & B(M,\varepsilon); \\ \mathbb{R}^n \setminus \mathrm{int} M_k & \subset & B(\mathbb{R}^n \setminus \mathrm{int} M, \varepsilon). \end{array}$$

Proof of Claim 4.3. Assume that the inclusion $M_k \subset B(M,\varepsilon)$ does not hold for k large enough. Then, without loss of generality, we assume that there is a sequence (x_k) in $\mathbb{R}^n$ such that, for all k, $x_k \in M_k$ and $d(x_k, M) \geq \varepsilon$. If $\{x | d(x, M) = \varepsilon\} \in M_k$: $S(M,\varepsilon) \cap M_k \neq \emptyset$, we let $y_k \in S(M,\varepsilon) \cap M_k$. If $S(M,\varepsilon) \cap M_k = \emptyset$, there is $y_k \in \mathrm{bd} M_k$ such that $d(y_k, M) > \varepsilon$. Indeed, let $x \in S(M,\varepsilon)$ such that $\|x_k - x\| = d(x_k, B(M,\varepsilon))$. Then $x \notin M_k$, $x_k \in M_k$ and hence there is $y_k \in (x, x_k] \cap \mathrm{bd} M_k$, and $d(y_k, B(M,\varepsilon)) > 0$; hence $d(y_k, M) > \varepsilon$. Then the sequence (y_k) is in the compact set $S(M,\varepsilon) \cup K$ (since $S(M,\varepsilon) \subset S(\mathrm{bd}M,\varepsilon)$, which is compact, and since $\mathrm{bd} M_k \subset K$). We may assume without any loss of generality that it converges to some $x \in S(M,\varepsilon) \cup K$. Then $d(x, M) \geq \varepsilon$, but, since $y_k \in M_k$ for all k, and since $M = \limsup M_k$, we get that $x \in M$, which is a contradiction. To get the second inclusion, apply the first result to the sets $\mathbb{R}^n \setminus \mathrm{int} M$ and $\mathbb{R}^n \setminus \mathrm{int} M_k$, noticing that assertion (int) (see Lemma 4.3) implies that $\mathbb{R}^n \setminus \mathrm{int} M = \limsup(\mathbb{R}^n \setminus \mathrm{int} M_k)$, that $\mathrm{bd}(\mathbb{R}^n \setminus \mathrm{int} M) = \overline{\mathrm{int} M} \setminus \mathrm{int} M \subset \mathrm{bd} M$, and that $\mathrm{bd}(\mathbb{R}^n \setminus \mathrm{int} M_k) = \overline{\mathrm{int} M_k} \setminus \mathrm{int} M_k \subset \mathrm{bd} M_k \subset K$. $\square$

Proof of (b). We let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map satisfying the conclusions of the following lemma, which is a slightly different version of Lemma 3.1 of Bonnisseau–Cornet [2] (its proof is left to the reader). Then, in view of Lemma 4.2, if we choose $\varepsilon < r'$, assertion (b) follows.

LEMMA 4.4. *There is $r' > 0$ and a bounded locally Lipschitzian map $F : \mathbb{R}^n \to \mathbb{R}^n$, such that*

$$\forall\ x \in B(\mathrm{bd}M, r'),\ \forall\ y \in G(x),\ (F(x)|y) > r'.$$

Proof of (c). We let $f : \mathbb{R}^n \to \mathbb{R}$ be a quasi-smooth representation of M satisfying the conclusions of Theorem 3.1, i.e., such that $f^{-1}([-\varepsilon_0, \varepsilon_0])$ is compact for some $\varepsilon_0 > 0$. Then from Lemma 4.4, since $\partial f(x) \subset N_M(x)$ for all $x \in \mathrm{bd} M$, since the correspondence ∂f is u.s.c., with nonempty convex compact values, and since $\mathrm{bd} M$ is compact, there is $r'' > 0$ such that

$$(F(x)|y) > r'' \quad \forall\ x \in \mathrm{bd}M \text{ and } \forall\ y \in \partial f(x).$$

Then assertion (c) holds, if we choose $\varepsilon > 0$ given by the following claim. $\square$

CLAIM 4.4. *Let f be a quasi-smooth representation of an epi-Lipschitzian subset $M \subset \mathbb{R}^n$ such that $f^{-1}([-r, r])$ is compact for some $r > 0$, and such that $(F(x)|y) > r$ for all $x \in \mathrm{bd} M$ and for all $y \in \partial f(x)$. Then there are two real numbers $\varepsilon > 0$ and $\alpha > 0$ such that*

(i) $B(\mathrm{bd}M, \varepsilon) \subset f^{-1}([-\alpha, \alpha]) \subset B(\mathrm{bd}M, r)$;

(ii) $\big(\partial f(\varphi(t, x)) | (\partial\varphi/\partial t)(t, x)\big) \subset [r, +\infty)$ $\forall\ x \in B(\mathrm{bd}M, r)$ *and* $\forall\ t \in \mathbb{R}$ *such that* $|f(\varphi(t, x))| \leq \alpha$;

(iii) *the function* $f \circ \varphi(., x)$ *is strictly increasing on* $\{t \in \mathbb{R} |\ |f(\varphi(t, x))| < \alpha\}$ $\forall\ x \in B(\mathrm{bd}M, r)$;

(iv) *there are t and t' in $\mathbb{R}$ such that* $f(\varphi(t, x)) \leq -\alpha$ *and* $f(\varphi(t', x)) \geq \alpha$ $\forall\ x \in f^{-1}([-\alpha, \alpha])$,

[12]Note that these inclusions imply that $\delta(M, M_k) \leq \varepsilon$ and that $\delta(\mathrm{bd}M, \mathrm{bd}M_k) \leq \varepsilon$, defining $\delta(X, Y) = \max\{\sup_{x\in X} d(x, Y), \sup_{y\in Y} d(y, X)\}$ if X and Y are two subsets of $\mathbb{R}^n$ (not necessarily nonempty compact), hence that $\delta(M, M_k) \to 0$ and that $\delta(\mathrm{bd}M, \mathrm{bd}M_k) \to 0$. Conversely, the assumption that $\delta(M, M_k) \in \mathbb{R}$ and converges to 0 implies that $M = \liminf M_k = \limsup M_k$.

recalling that $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is the flow of the differential equation* (E) $\dot{x}(t) = F(x(t))$, $x(0) = x$, *where* $\varphi(.,x)$ *is defined on* $\mathbb{R}$.

Proof of Claim 4.4. Since the correspondence ∂f is u.s.c. with compact convex values and since the map F is continuous, the correspondence H defined by $H(x) = (F(x)|\partial f(x))$ is u.s.c., with compact convex values. Then, from Lemma 3.1, there is $\beta > 0$ such that $B(\mathrm{bd}M, \beta) \subset U$ and such that

$$(F(x)|y) \quad > \quad r \qquad \text{for all } x \in B(\mathrm{bd}M, \beta) \text{ and for all } y \in \partial f(x).$$

Then there is $\alpha > 0$ such that $f^{-1}([-\alpha,\alpha]) \subset B(\mathrm{bd}M,\beta)$ (since $B(\mathrm{bd}M,\beta)$ is an open set containing the intersection of compact sets $\cap_{\alpha\in(0,\varepsilon_0]} f^{-1}([-\alpha,\alpha))$. Without any loss of generality, we may assume that $f^{-1}([-\alpha,\alpha]) \subset B(\mathrm{bd}M, r)$. Since $f^{-1}((-\alpha,\alpha))$ is an open set containing the intersection of compact sets $\cap_{\epsilon>0}\overline{B}(\mathrm{bd}M,\epsilon)$, there is $\varepsilon > 0$ such that $B(\mathrm{bd}M,\varepsilon) \subset f^{-1}([-\alpha,\alpha])$, which proves (i). Let $x \in B(\mathrm{bd}M, r)$. We define the function $h_x : \mathbb{R} \to \mathbb{R}$ by $h_x(t) = f(\varphi(t,x))$. Then, from Clarke [4],

$$\partial h_x(t) \subset \Big(\partial f(\varphi(t,x))|(\partial\varphi/\partial t)(t,x)\Big),$$

i.e., $\partial h_x(t) \subset \Big(\partial f(\varphi(t,x))|F(\varphi(t,x))\Big)$, which is minorized by r if $|f(\varphi(t,x))| < \alpha$; this proves (ii). Hence from the mean-value theorem, the function $f \circ \varphi(.,x) = h_x$ is strictly increasing on $\{t \in \mathbb{R}|\ |f(\varphi(t,x))| < \alpha\}$, which proves (iii). If we additionally assume that $x \in f^{-1}([-\alpha,\alpha])$, then (see, for example, Hirsch and Smale [9]; the fact can also be proved directly) $\varphi(t,x) \notin f^{-1}([-\alpha,\alpha])$ when t is large enough; hence, from (ii), $f(\varphi(t,x)) \geq \alpha$ when $t \to +\infty$. In the same way, $f(\varphi(t,x)) \leq -\alpha$ when $t \to -\infty$. ◻

4.3. The sets M and M_k are lipeomorphic. In view of Propositions 4.1 and 4.2, the proof of Theorem 2.3 is finished if we prove the next proposition.

PROPOSITION 4.3. *Let M and N be two closed subsets of $\mathbb{R}^n$ admitting a Lipschitzian transverse field in the sense of* (T)*. Then M and N are epi-Lipschitzian and lipeomorphic.*

Remark 4.2. One could reduce Proposition 4.3 to the smooth case. Indeed, since $\mathrm{bd}M$ and $\mathrm{bd}N$ are compact, there are two smooth normal approximations (M_k) and (N_k) of M and N, respectively. Then, for k large enough, there is a Lipschitzian transverse field between the two sets M_k and N_k in the sense of (T). But the proof would be identical. Proposition 4.3 can be proved directly, without using the notion of normal approximation, as we do in the following.

Proof of Proposition 4.3. The set M is clearly epi-Lipschitzian, since the transversality condition (T) implies that, for every $x \in \mathrm{bd}M$, $\mathrm{int}T_M(x) \neq \emptyset$, hence that $N_M(x)$ is pointed. Similarly, the set N is also epi-Lipschitzian. Let f_M (resp., f_N) be an inequality representation of M (resp., N) satisfying the conclusions of Theorem 3.1, i.e., such that $f_M^{-1}([-\varepsilon_0,\varepsilon_0])$ (resp., $f_N^{-1}([-\varepsilon_0,\varepsilon_0])$) is compact for some $\varepsilon_0 > 0$. The following lemma is a different version of Lemma 3.2 of Bonnisseau–Cornet [2] that we prove for the sake of completeness.

LEMMA 4.5. *There is a real number $\beta > 0$ and two Lipschitzian functions τ and θ, defined from $U \times [-\beta,\beta]$ to $\mathbb{R}^n$, such that*

$$\text{(i)} \quad \Big\{x \in \mathbb{R}^n|\min\{f_M(x), f_N(x)\} \leq \beta, \max\{f_M(x), f_N(x)\} \geq -\beta\Big\} \subset U$$

and such that, for all $(x,\delta,t) \in U \times [-\beta,\beta] \times \mathbb{R}$, then

(ii) $f_M(\varphi(x,t)) = \delta \Leftrightarrow t = \tau(x,\delta)$;
(iii) $f_N(\varphi(x,t)) = \delta \Leftrightarrow t = \theta(x,\delta)$.

Proof of Lemma 4.5. Note that, without any loss of generality, we only need to prove assertions (i) and (ii). Since the set

$$\Big\{x \in \mathbb{R}^n \,|\, \min\{f_M(x), f_N(x)\} \leq 0, \max\{f_M(x), f_N(x)\} \geq 0\Big\}$$

is equal to the set $[M \setminus \mathrm{int}N] \cup [N \setminus \mathrm{int}M]$, which is included in the set U, we get the inclusion

$$\cap_{\beta\in(0,\varepsilon_0]}\Big\{x \in \mathbb{R}^n \,|\, \min\{f_M(x), f_N(x)\} \leq \beta, \max\{f_M(x), f_N(x)\} \geq -\beta\Big\} \subset U.$$

This implies assertion (i) for some $\beta > 0$ (since we have an intersection of compact sets included in U). Then from the assumption (T), (b), there is $r > 0$ such that $(F(x)|y) > r$ for all $x \in \mathrm{bd}M$ and for all $y \in \partial f_M(x)$. We may assume that β is small enough and we thus satisfy the conclusions of Claim 4.4, given the function f_M. Then $f_M^{-1}([-\beta,\beta]) \subset U$, and the function $f_M \circ \varphi(.,x)$ is strictly increasing on $\{t \in \mathbb{R} |\, |f_M(\varphi(t,x))| < \beta\}$ for every $x \in \mathbb{R}^n$. Let us now consider $x \in U$. From the assumption (T), (c), there is $t \in \mathbb{R}$ such that $\varphi(x,t) \in \mathrm{bd}M$, i.e., $f_M(\varphi(x,t)) = 0$. Let $(\delta,t) \in [-\beta,\beta] \times \mathbb{R}$. Then there is one and only one $t' \in \mathbb{R}$ such that $f_M(\varphi(t',x)) = \delta$. We let $t' = \tau(x,\delta)$. Let us now prove that the function τ is Lipschitzian. We let

$$\Omega = \{(x,t,\delta) \in \Omega \times \mathbb{R} |\, |f_M(\varphi(t,x))| < \beta\},$$

and we define $H : \Omega \to \mathbb{R}$ by $H(x,t,\delta) = f_M(\varphi(t,x)) - \delta$. The function H is clearly Lipschitzian. The fact that τ is Lipschitzian around some (x,δ) is a direct consequence of the implicit function theorem [4, p. 255], if we prove that $t^* \neq 0$, for every element $(x^*,t^*,\delta^*) \in \partial H(x,t,\delta)$, where $t \in \mathbb{R}$ satisfies $H(x,t,\delta) = 0$. In other words, if $\pi_t : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the projection defined by $\pi_t(x,t,\delta) = t$, if $0 \notin \pi_t[\partial H(x,t,\delta)]$. From Clarke [4, Proposition 2.6.2], we get that

$$\pi_t[\partial H(x,t,\delta)] \subset \Big(\partial f_M(\varphi(t,x))|F(\varphi(t,x))\Big),$$

which is minorized by β if we chose β small enough (Claim 4.4). □

We then get the following property on the functions τ and θ.

Claim 4.5. *For all* $(x,\delta,t) \in U \times [-\beta,\beta] \times \mathbb{R}$

(i) $f_M(\varphi(x,t)) \geq \delta \Leftrightarrow t \geq \tau(x,\delta)$;
(ii) $f_N(\varphi(x,t)) \geq \delta \Leftrightarrow t \geq \theta(x,\delta)$;

If $y \in U$ *is such that* $\varphi(x,t) = y$ *for some* t, *then* $\tau(x,\delta) = \tau(y,\delta) + t$ *and* $\theta(x,\delta) = \theta(y,\delta) + t$ *for all* δ.

Proof of Claim 4.5. The proof comes from Lemma 4.5 and from the Cauchy–Lipschitz theorem for φ, since $f_M \circ \varphi(x,.)$ is strictly increasing on $\{t \in \mathbb{R} | \,|f_M(\varphi(t,x))| < \beta\}$. □

We are now able to build the lipeomorphism between M and N. Let us define the map $h : M \to N$ by

$$\begin{aligned} h(x) &= x && \text{if } x \in \mathrm{int}(M_{-\beta} \cap N_{-\beta}),\\ h(x) &= \varphi\Big(x, i_x(\tau_x - \theta_x)/(\tau_x - i_x)\Big) && \text{if } x \in M \setminus \mathrm{int}(M_{-\beta} \cap N_{-\beta}), \end{aligned}$$

where we let $M_{-\beta} = \{x \in \mathbb{R}^n | f_M(x) \leq -\beta\}$, $N_{-\beta} = \{x \in \mathbb{R}^n | f_N(x) \leq -\beta\}$, $\tau_x = \tau(x,0)$, $\theta_x = \theta(x,0)$, and $i_x = \inf\{\tau(x,-\beta), \theta(x,-\beta)\}$. Let $x \in M \setminus \mathrm{int}(M_{-\beta} \cap N_{-\beta})$; then $f_M(x) \leq 0$ and $\max\{f_M(x), f_N(x)\} \geq -\beta$; hence $x \in U$ from Lemma 4.5. Hence the map h is well defined. The map h has its values in N. Indeed, let $x \in M \setminus \mathrm{int}(M_{-\beta} \cap N_{-\beta})$. Then, recalling that $x = \varphi(x,0)$,

$$\begin{aligned} & f_M[\varphi(x,0)] = f_M(x) \geq -\beta = f_M[\varphi(x,\tau(x,-\beta))], \\ \text{or}\quad & f_N[\varphi(x,0)] = f_N(x) \geq -\beta = f_N[\varphi(x,\theta(x,-\beta))]; \end{aligned}$$

hence, from Claim 4.5, and $0 \geq \tau(x,-\beta)$ or $0 \geq \theta(x,-\beta)$, which implies that $i_x \leq 0$. The same claim implies that $i_x < \tau_x$ (since $f_M[\varphi(x,\tau(x,-\beta))] < f_M[\varphi(x,\tau_x)]$), $i_x < \theta_x$ (since $f_N[\varphi(x,\theta(x,-\beta))] < f_N[\varphi(x,\theta_x)]$), and $\tau_x \geq 0$ (since $f_M[\varphi(x,0)] = f_M(x) \leq 0 = f_M[\varphi(x,\tau_x)]$). Then we get that $i_x(\tau_x - \theta_x)/(\tau_x - i_x) \leq \theta_x$, which proves that $f_N(h(x)) \leq 0$, i.e., $h(x) \in N$.

CLAIM 4.6. *The map h is locally Lipschitzian.*

Proof of Claim 4.6. The map h is clearly Lipschitzian on $\mathrm{int}(M_{-\beta} \cap N_{-\beta})$. From Lemma 4.5, the map $x \mapsto \inf\{\tau(x,-\beta), \theta(x,-\beta)\}$ is Lipschitzian and $\tau(x,0) - \inf\{\tau(x,-\beta), \theta(x,-\beta)\} \neq 0$ for all $x \in M \setminus (M_{-\beta} \cap N_{-\beta})$. Then clearly h is Lipschitzian on $M \setminus \mathrm{int}(M_{-\beta} \cap N_{-\beta})$. If $x \in \mathrm{bd}(M_{-\beta} \cap N_{-\beta})$, then $f_M(x) = -\beta$ or $f_N(x) = -\beta$; besides, $f_M(x) \leq -\beta$ and $f_N(x) \leq -\beta$. This implies that $\tau(x,-\beta) = 0$ or $\theta(x,-\beta) = 0$, and that $\tau(x,-\beta) \geq 0$ and $\theta(x,-\beta) \geq 0$. Hence $\inf\{\tau(x,-\beta), \theta(x,-\beta)\} = 0$ and $h(x) = x$. This proves that h is Lipschitzian on $M_{-\beta} \cap N_{-\beta}$. Hence h is Lipschitzian on M. $\square$

CLAIM 4.7. *The map h is one to one, and h^{-1} is Lipschitzian.*

Proof of Claim 4.7. Let us define the map $k : N \to M$ by

$$\begin{aligned} k(x) &= x & & \text{if } x \in \mathrm{int}(M_{-\beta} \cap N_{-\beta}); \\ k(x) &= \varphi\Big(x, i_x(\theta_x - \tau_x)/(\theta_x - i_x)\Big) & & \text{if } x \in N \setminus \mathrm{int}(M_{-\beta} \cap N_{-\beta}). \end{aligned}$$

We let the reader check that k has its values in M and that it is Lipschitzian. Let us now prove that $h \circ k = id_N$. Let us consider $x \in M$. We let $y = k(x)$; then $i_y = i_x - t$, $\tau_y = \tau_x - t$, and $\theta_y = \theta_x - t$, where $t = i_x(\theta_x - \tau_x)/(\theta_x - i_x)$. Hence $i_y(\tau_y - \theta_y)/(\tau_y - i_y) = i_x(\tau_x - \theta_x)/(\theta_x - i_x)$, and hence $h(y) = \varphi(y, i_y(\tau_y - \theta_y)/(\tau_y - i_y)) = x$. The fact that $k \circ h = id_M$ goes in the same way. $\square$

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REFERENCES

[1] J. BENOIST, *Approximation and regularization of arbitrary sets in finite dimension*, Set-Valued Anal., 2 (1994), pp. 95–115.

[2] J.-M. BONNISSEAU AND B. CORNET, *Fixed-point theorem and Morse's lemma for Lipschitzian functions*, J. Math. Anal. Appl., 146 (1990), pp. 318–322.

[3] A. CELLINA, *A theorem on the approximation of compact multi-valued mappings*, Atti Accad. Naz. Lincei Cl. Sci. Fis., Mat. Natur. Rend. (8), 47 (1969), pp. 429–433.

[4] F. CLARKE, *Optimization and Nonsmooth Analysis*, John Wiley, New York, 1983. Reprinted in Classics Appl. Math. 5 (1989), SIAM, Philadelphia, PA, 1989.

[5] B. CORNET, *Regularity properties of open tangent cones*, Math. Prog. Study, 30 (1987), pp. 17–33.

[6] B. CORNET AND M.-O. CZARNECKI, *Smooth representations of epi-Lipschitzian subsets of* $\mathbb{R}^n$, Nonlinear Anal. (1998), to appear. French abridged version in C. R. Acad. Sci. Sér. 1, 325 (1997), pp. 475–480.

[7] B. Cornet and M.-O. Czarnecki, *Existence of (generalized) equilibria on an epi-Lipschitzian domain: A necessary and sufficient condition*, Cahier Eco-Maths 95-55, Université de Paris 1, Paris, France, 1995.
[8] P. Doktor, *Approximation of domains with Lipschitzian boundary*, Casopis Pest. Mat., 101 (1976), pp. 237–255.
[9] M.W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1960.
[10] A.D. Ioffe, *Approximate subdifferentials and applications* I: *The finite dimensional theory*, Trans. Amer. Math. Soc., 281 (1984), pp. 389–416.
[11] A.Y. Kruger and B.S. Mordukhovich, *Generalized normals and derivatives, and necessary optimality conditions in nondifferentiable programming, Depon. VINITI*, Nos. 408–80 and 494–80, Moscow, 1980 (in Russian).
[12] K. Kuratowski, *Topologie*, Państwowe Wydawnictwo Naukowe, Warszawa, 1958.
[13] U. Massari and L. Pepe, *Sull'approssimazione degli aperti lipschitziani di* $\mathbb{R}^n$ *con varietà differenziabili*, Boll. Un. Mat. Ital., Ser. 4, 10 (1974), pp. 532–544.
[14] B.S. Mordukhovich, *Maximum principle in the problem of time optimal response with non-smooth constraints*, J. Appl. Math. Mech. 40 (1976), pp. 960–969.
[15] J. Nečas, *On type* $\mathcal{M}$ *domains*, Czechoslovak Math. J., 12 (1962), pp. 274–287 (in Russian).
[16] R.T. Rockafellar, *Clarke's tangent cones and the boundaries of closed sets in* $\mathbb{R}^n$, Nonlinear Anal., 3 (1979), pp. 145–154.